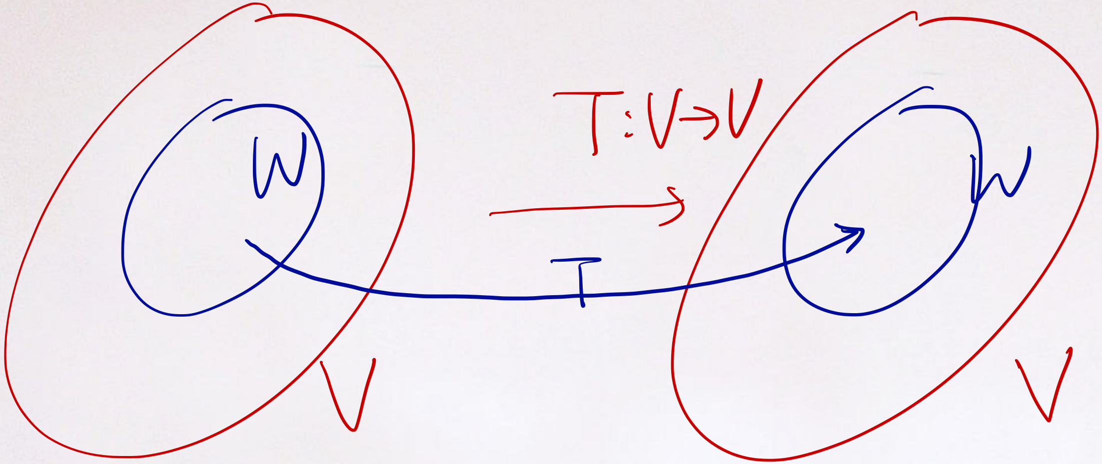
Lecture 15:

<u>Definition:</u> Let T be a linear operator on a vector space V.

A subspace $W \subset V$ is called T-invariant if $T(W) \subseteq W$.

That is, $T(\vec{w}) \in W$ for $\forall \vec{w} \in W$.



Def: $T|_W : W \to W$ defined by $T|_W(\vec{w}) = T(\vec{w})$

$f_{T|_W}(t)$ divides $f_T(t)$

Remark: Let $T : V \to V$ be a linear operator on a finite-dim vector space V, and let $W \subset V$ be a T-invariant subspace.

Then, the restriction of T to W, denote it by $T|_W : W \to W$, is well-defined and linear.

Proposition: $f_{T|_W}(t)$ divides $f_T(t)$.

Proof: Choose an ordered basis $\gamma = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for W and extend it to an ordered basis $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V. Then:

$$[T]_\beta = \begin{pmatrix} | & | & \\ [\overset{\in W}{T(\vec{v}_1)}]_\beta & [T(\vec{v}_2)]_\beta & \cdots \\ | & | & \end{pmatrix} = \begin{array}{cc} \begin{matrix} \overbrace{\qquad\qquad}^{k} & \qquad \end{matrix} & \\ \begin{pmatrix} [T|_W]_\gamma & B \\ O & C \end{pmatrix} & \begin{matrix} \}\,k \\ \\ \end{matrix} \end{array}$$

$$f_T(t) = \det\left[\begin{pmatrix} [T_W]_\gamma & B \\ O & C \end{pmatrix} - t\,I\right]$$

$$= \det\begin{pmatrix} [T_W]_\gamma - t\,I_k & B \\ O & C - t\,I_{n-k} \end{pmatrix}$$

$$= \det\left([T_W]_\gamma - t\,I_k\right) \underbrace{\det\left(C - t\,I_{n-k}\right)}_{g(t)}$$

$$= f_{T_W}(t)\; g(t)$$

$\therefore\; f_{T_W}(t)$ divides $f_T(t)$

Theorem: Let $T: V \to V$ be a linear operator on a finite-dim vector space V and let $W \subset V$ be T-cyclic subspace of V generated by $\vec{v} \neq \vec{0} \in V$. ($W = \text{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots\}$)

Let $k = \dim(W)$. Then:

(a) $\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is a basis for W

(b) If $a_0\vec{v} + a_1 T(\vec{v}) + a_2 T^2(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$,

then the characteristic polynomial of $T|_W$ is:

$$f_{T_W}(t) = (-1)^k \left(a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1} t^{k-1} + t^k\right)$$

Proof: (a) Since $\vec{v} \neq \vec{0}$, then $\{\vec{v}\}$ is linearly independent.

Let j be the largest +ve integer s.t.

$$\beta = \{\vec{v}, T(\vec{v}), \ldots, T^{j-1}(\vec{v})\} \text{ is linearly independent.}$$

Such j exists because V is finite-dim.

Let $Z = \mathrm{span}(\beta)$. $\therefore$ $Z \subset W$

Then, $\underbrace{\beta}_{\text{L.I.}} \cup T^j(\vec{v})$ is linearly dependent. $\therefore$ $T^j(\vec{v}) \in \underbrace{\mathrm{span}(\beta)}_{= Z}$

$\therefore$ $T^j(\vec{v}) \in Z$.

Now, let $\vec{w} \in Z$. Then $\exists\, b_0, b_1, \ldots, b_{j-1} \in F$ s.t.

$$\vec{w} = b_0 \vec{v} + b_1 T(\vec{v}) + \cdots + b_{j-1} T^{j-1}(\vec{v})$$

$$T(\vec{w}) = b_0 \underbrace{T(\vec{v})}_{\in Z} + b_1 \underbrace{T^2(\vec{v})}_{\in Z} + \cdots + b_{j-2} \underbrace{T^{j-1}(\vec{v})}_{\in Z} + b_{j-1} \underbrace{T^j(\vec{v})}_{\in Z} \in Z$$

$\therefore$ If $\vec{w} \in Z$, then $T(\vec{w}) \in Z$.

$\therefore$ Z is T-invariant subspace containing $\vec{v}$.

$\therefore$ $\underbrace{W}_{\substack{\parallel \\ \text{T-cyclic subspace} \\ \text{containing } \vec{v}}} \subset Z$. ($\because$ W is smallest T-invariant subspace containing $\vec{v}$)

$\therefore$ $W = Z = \text{span}(\overbrace{\beta}^{\text{L.I.}})$

$\therefore$ β is a basis of W and $j = k$.

(b) By (a), $\beta = \{\vec{v}, T(\vec{v}), \ldots, T^{k-1}(\vec{v})\}$ is an ordered basis for W.

Let $a_0, \ldots, a_{k-1} \in F$ s.t.

$$a_0\vec{v} + a_1 T(\vec{v}) + \cdots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$$

$$\Rightarrow T^k(\vec{v}) = -a_0\vec{v} - a_1 T(\vec{v}) - \cdots - a_{k-1} T^{k-1}(\vec{v}).$$

Then: $$[T]_\beta = \begin{pmatrix} | & | & & | \\ [T(\vec{v})]_\beta & [T(T(\vec{v}))]_\beta & \cdots & [T^k(\vec{v})]_\beta \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{pmatrix}$$

$$f_{T_W}(t) \overset{\text{"}}{=} \det\left(\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & 0 & \cdots\ 0 & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & & 1 \quad 0 & -a_{k-1} \end{pmatrix} - t I_k \right)$$

$$\parallel$$

$$(-1)^k (a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k) \quad (HW)$$

Theorem: (Cayley-Hamilton) Let T be a linear operator on a finite-dim. vector space V and let $f(t) = f_T(t)$ be a char poly of T. Then: $f(T) =$ zero transformation.

(Char poly "kills" the linear operator T)

Remark:

$$f(t) = a_0 1 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \dots + a_n T^n$$

Proof: We want to show $f(T)(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.

$f(T)(\vec{0}) = \vec{0}$ ($\because$ $f(T)$ is linear)

So, suppose $\vec{v} \neq \vec{0}$. Let W = T-cyclic subspace generated by $\vec{v}$.

Let $k = \dim(W)$

By Thm we have shown last time:

$\exists\ a_0, a_1, \dots, a_{k-1} \in F$ such that:

$$\begin{cases} \bullet\ a_0 \vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1} T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \bullet\ g(t) \stackrel{\text{def}}{=} f_{T|_W} = (-1)^k \left(a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k \right) \end{cases}$$

$$\begin{cases} \cdot\ a_0\vec{v} + a_1 T(\vec{v}) + \dots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0} \\ \cdot\ g(t) \overset{\text{def}}{=} f_{T|_W} = (-1)^k(a_0 + a_1 t + \dots + a_{k-1}t^{k-1} + t^k) \end{cases}$$

$$\Downarrow$$

$$g(T)(\vec{v}) = \vec{0}$$

Now, $g(t) \mid f(t)$ $\overset{\text{implies}}{\rightsquigarrow}$ $\exists\, q(t)$ s.t. $f(t) = q(t)g(t)$

$$f(T) = q(T)g(T)$$

$$q(T)\circ g(T)$$

$$\therefore\ f(T)(\vec{v}) = q(T)\circ \underbrace{g(T)(\vec{v})}_{\vec{0}} = \vec{0}$$

Corollary: Let $A \in M_{n \times n}(F)$ and $f(t)$ be its char. poly. Then: $f(A) = O$, the zero matrix.

Tensor Product Space

Definition: A map $B : X \times Y \to Z$ (X, Y, Z are vector spaces over F) is a bilinear map if:

$$B(a\vec{u} + b\vec{v}\,,\ c\vec{s} + d\vec{t}) = ac\,B(\vec{u},\vec{s}) + ad\,B(\vec{u},t) + bc\,B(\vec{v},\vec{s}) + bd\,B(\vec{v},t)$$

where $\vec{u}, \vec{v} \in X$, $\vec{s}, t \in Y$, $a, b, c, d \in F$.

Definition: Let V_1, V_2 be vector spaces over F. A pair (Y, μ), where Y is a vector space over F and $\mu : V_1 \times V_2 \to Y$ is a bi-linear map, is called the tensor product of V_1 and V_2 if the following condition holds:

(*) whenever β_1 is a basis for V_1 and β_2 is a basis for V_2, then:

$$\mu(\beta_1 \times \beta_2) \stackrel{\text{def}}{:=} \{\mu(\vec{x}_1, \vec{x}_2) : \vec{x}_1 \in \beta_1, \vec{x}_2 \in \beta_2\} \text{ is a}$$

basis for Y.

Remark: We write $V_1 \otimes V_2$ for Y.

We write $\vec{x}_1 \otimes \vec{x}_2$ for $\mu(\vec{x}_1, \vec{x}_2)$.